# CORING STRUCTURES AND HILBERT $C^*$-MODULES (Recent Aspects of C^*-algebras)



O'uchi, Moto



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# CORING STRUCTURES AND HILBERT $C^*$-MODULES

大阪女子大学理学部　大内本夫 (MOTO O'UCHI)
OSAKA WOMEN'S UNIVERSITY

## 1. Introduction

S. Baaj and G. Skandalis [1] introduced the notion of multiplicative unitaries and they studied Hopf $C^*$-algebras associated with them. J. M. Vallin introduced the notion of pseudo-multiplicative unitaries and algebraic structures associated with them ([11], [12]). M. Enock and Vallin [2] studied pseudo-multiplictive unitaries and quantum groupoids associated with inclusions of von Neumann algebras. The author introduced a notion of multiplicative unitary operators (MUO) on Hilbert $C^*$-modules ([8], see also [6] and [7]). It is interesting to study natural algebraic structures associated with MUO's. In this note, we will show the relation btween MUO's and coring structures on Hilbert $C^*$-modules. Coring structures were introduced by M. Sweedler [10] in the purely algebraic framework. Y. Watatani [13] showed that incusions of $C^*$-algebras give natural coring structures in the framework of his index theory. In this note, we introduce notions of coring structures on Hilbert $C^*$-modules and study coring structures associated with MUO's. In Sections 2 and 3, we study coring structures associated with MUO's arising from groupoids and inclusions of $C^*$-algebras of inedx finite type in the sense of Watatani. In the case of groupoids, the base algebras are commutative. In the case of inclusions of

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$C^*$-algebras of index finite type, we do not know any concrete examples of MUO's on infinite-dimensional Hilbert $C^*$-modules. Therefore it is interesting to study concrete examples of MUO's and the associated coring structures such that the base algebras are not commutative and the Hilbert $C^*$-modules are infinite-dimensional. In the last section, we study an MUO and the associatd coring structures on the Hilbert $C^*$-module of compact operators. In this case, the Hilbert $C^*$-module is infinite-dimensional and the base algebra is the $C^*$-algebra of compact operators.

## 2. Preliminaries

2.1. **Multiplicative operators on Hilbert $C^*$-modules.** Let $A$ be a $C^*$-algebra, let $E$ be a Hilbert $A$-module and let $\phi$ and $\psi$ be $*$-homomorphisms of $A$ to $\mathcal{L}_A(E)$. We assume that $\phi$ and $\psi$ commute, that is, $\phi(a)\psi(b) = \psi(b)\phi(a)$ for all $a$, $b \in A$. We define a $*$-homomorphism $\iota \otimes_\phi \psi$ of $A$ to $\mathcal{L}_A(E \otimes_\phi E)$ by $(\iota \otimes_\phi \psi)(a) = I \otimes_\phi \psi(a)$. and define a $*$-homomorphism $\iota \otimes_\psi \phi$ of $A$ to $\mathcal{L}_A(E \otimes_\psi E)$ by $(\iota \otimes_\psi \phi)(a) = I \otimes_\psi \phi(a)$. Let $W$ be an operator in $\mathcal{L}_A(E \otimes_\psi E, E \otimes_\phi E)$. We assume that $W$ satisfies the following equations;

$$W(\iota \otimes_\psi \phi)(a) = (\phi \otimes_\phi \iota)(a)W, \tag{2.1}$$

$$W(\psi \otimes_\psi \iota)(a) = (\iota \otimes_\phi \psi)(a)W, \tag{2.2}$$

$$W(\phi \otimes_\psi \iota)(a) = (\psi \otimes_\phi \iota)(a)W \tag{2.3}$$

for all $a \in A$. Then we can define following operators;

$$W \otimes_\psi I \in \mathcal{L}_A(E \otimes_\psi E \otimes_\psi E, E \otimes_\phi E \otimes_\psi E),$$

$$I \otimes_{\phi \otimes \iota} W \in \mathcal{L}_A(E \otimes_\phi E \otimes_\psi E, E \otimes_\psi E \otimes_\phi E),$$

$$W \otimes_\phi I \in \mathcal{L}_A(E \otimes_\psi E \otimes_\phi E, E \otimes_\phi E \otimes_\phi E),$$

$$I \otimes_{\psi \otimes \iota} W \in \mathcal{L}_A(E \otimes_\psi E \otimes_\psi E, E \otimes_{\iota \otimes \psi} (E \otimes_\phi E)),$$

$$I \otimes_{\iota \otimes \phi} W \in \mathcal{L}_A(E \otimes_{\iota \otimes \phi} (E \otimes_\psi E), E \otimes_\phi E \otimes_\phi E).$$

Since $\phi$ and $\psi$ commute, there exists an isomorphism $\Sigma_{12}$ of $E \otimes_{\iota\otimes\psi} (E \otimes_\phi E)$ onto $E \otimes_{\iota\otimes\phi} (E \otimes_\psi E)$ as Hilbert $A$-modules such that, for $x_i \in E$ $(i = 1, 2, 3)$,

$$\Sigma_{12}(x_1 \otimes (x_2 \otimes x_3)) = x_2 \otimes (x_1 \otimes x_3).$$

**Definition 2.1** ([8]). Let $W$ be an element of $\mathcal{L}_A(E \otimes_\psi E, E \otimes_\phi E)$. Assume that $W$ satisfies the equations (2.1), (2.2) and (2.3). An operator $W$ is said to be multiplicative if it satisfies the pentagonal equation

$$(W \otimes_\phi I)(I \otimes_{\phi\otimes\iota} W)(W \otimes_\psi I) = (I \otimes_{\iota\otimes\phi} W)\Sigma_{12}(I \otimes_{\psi\otimes\iota} W). \tag{2.4}$$

**Example 2.2.** Suppose that $A = \mathbb{C}$. Then $E = H$ is a usual Hilbert space and $\mathcal{L}_{\mathbb{C}}(E) = \mathcal{L}(H)$ is the $C^*$-albebra of bounded linear operators on $H$. Let $\phi = \psi = id$, where $id(\lambda) = \lambda I_H$ for $\lambda \in \mathbb{C}$. Then $E \otimes_{id} E$ is the usual tensor product $H \otimes H$. Let $\Sigma \in \mathcal{L}(H \otimes H)$ be the flip, that is, $\Sigma(\xi \otimes \eta) = \eta \otimes \xi$. Let $W$ be an element of $\mathcal{L}(H \otimes H)$. Then the pentagonal equation (2.4) has the following form:

$$(W \otimes I)(I \otimes W)(W \otimes I) = (I \otimes W)(\Sigma \otimes I)(I \otimes W). \tag{2.5}$$

Defin an operator $\widetilde{W}$ by $\widetilde{W} = W\Sigma$. Then $W$ satisfies the pentagonal equation (2.5) if and only if $\widetilde{W}$ satisfies the usual pentagonal equation ;

$$\widetilde{W}_{12}\widetilde{W}_{13}\widetilde{W}_{23} = \widetilde{W}_{23}\widetilde{W}_{13}. \tag{2.6}$$

2.2. **Coproducts on Hilbert $C^*$-modules.** Let $E$ be a Hilbert $A$-module and $\phi$ be a $*$-homomorphism of $A$ to $\mathcal{L}_A(E)$.

**Definition 2.3.** Let $\delta$ be an operator in $\mathcal{L}_A(E, E \otimes_\phi E)$. We say that $\delta$ is a coproduct of $(E, \phi)$ if $\delta$ satisfies the following equations;

$$\delta\phi(a) = (\phi \otimes \iota)(a)\delta \qquad \text{for all } a \in A, \tag{2.7}$$

$$(\delta \otimes I_E)\delta = (I_E \otimes \delta)\delta. \tag{2.8}$$

Suppose that $\delta$ is a coproduct for $E$. For $\xi$, $\eta \in E$, we define a product $\xi\eta$ in $E$ by $\xi\eta = \delta^*(\xi \otimes_\phi \eta)$. It follows from (2.8) that this product is associative. Then $E$ is an algebra over $\mathbb{C}$. Note that we have $\|\xi\eta\| \leq \|\delta\|\|\xi\|\|\eta\|$.

2.3. **Coproducts associated with MUO's.** Let $E$ be a Hilbert $A$-module and let $\phi$ and $\psi$ be $*$-homomorphisms of $A$ to $\mathcal{L}_A(E)$ such that $\phi$ and $\psi$ commute. Let $W \in \mathcal{L}_A(E \otimes_\psi E, E \otimes_\phi E)$ be a multiplicative unitary operator (MUO).

For an element $\xi_0$ of $E$, we say that $\xi_0$ has Property E1 if it satisfies the following conditions;

(i) $W(\xi_0 \otimes_\psi \xi_0) = \xi_0 \otimes_\phi \xi_0$.

(ii) For every $\xi \in E$, there exists an element $\pi_{\xi_0}(\xi)$ of $\mathcal{L}_A(E)$ such that

$$< \eta, \pi_{\xi_0}(\xi)\zeta >=< W(\xi_0 \otimes_\psi \eta), \xi \otimes_\phi \zeta > \quad \text{for every } \eta,\, \zeta \in E.$$

Fix an element $\xi_0$ with Property E1. Define an operator $\delta = \delta_{\xi_0}$ in $\mathcal{L}_A(E, E \otimes_\phi E)$ by $\delta(\eta) = W(\xi_0 \otimes_\psi \eta)$. Then we have $\delta^*(\xi \otimes \eta) = \pi_{\xi_0}(\xi)\eta$. Since $W$ satisfies the pentagonal equation, $\delta$ is a coproduct of $(E, \phi)$.

For an element $\xi_0$ of $E$, we say that $\xi_0$ has Property E2 if it satisfies the following conditions;

(i) $W(\xi_0 \otimes_\psi \xi_0) = \xi_0 \otimes_\phi \xi_0$.

(ii) For every $\xi \in E$, there exists an element $\widehat{\pi}_{\xi_0}(\xi)$ of $\mathcal{L}_A(E)$ such that

$$< \eta, \widehat{\pi}_{\xi_0}(\xi)\zeta >=< W^*(\xi_0 \otimes_\phi \eta), \xi \otimes_\psi \zeta > \quad \text{for every } \eta,\, \zeta \in E.$$

Fix an element $\xi_0$ with Property E2. Define an operator $\widehat{\delta} = \widehat{\delta}_{\xi_0}$ in $\mathcal{L}_A(E, E \otimes_\psi E)$ by $\widehat{\delta}(\eta) = W^*(\xi_0 \otimes_\phi \eta)$. Since $W$ satisfies the pentagonal equation, $\widehat{\delta}$ is a coproduct of $(E, \psi)$.

## 3. Coring structures on Hilbert $C^*$-modules

Let $E$ be a Hilbert $A$-module and let $\phi$ be a $*$-homomorphism of $A$ to $\mathcal{L}_A(E)$. Note that $A$ itself is a Hilbert $A$-module with the $A$-valued inner product $< a, b >= a^*b$.

We denote by $i$ the $*$-homomorphism of $A$ to $\mathcal{L}_A(A)$ defined by $i(a)b = ab$. Then there exists a unitary operator $t$ in $\mathcal{L}_A(E \otimes_i A, E)$ defined by $t(\xi \otimes_i a) = \xi a$. If $\phi$ is non-degenerate, then there exists a unitary operator $t'$ in $\mathcal{L}_A(A \otimes_\phi E, E)$ such that $t'(a \otimes_\phi \xi) = \phi(a)\xi$.

**Definition 3.1.** Suppose that $\phi$ is non-degenerate. Let $\delta$ be a coproduct of $(E, \phi)$ and let $Q$ be an element of $\mathcal{L}_A(E, A)$, such that $Q\phi(a) = aQ$ for $a \in A$.

(1) We say that $(E, \phi, \delta, Q)$ is a right counital $A$-coring if it satisfies the following equation;

$$t(I_E \otimes_\phi Q)\delta = I_E.$$

Then $Q$ is called a right counit.

(2) We say that $(E, \phi, \delta, Q)$ is a left counital $A$-coring if it satisfies the following equation;

$$t'(Q \otimes_\phi I_E)\delta = I_E.$$

Then $Q$ is called a left counit.

(3) We say that $(E, \phi, \delta, Q)$ is a counital $A$-coring if $Q$ is a right and left counit. Then $Q$ is called a counit.

For $n \geq 2$, we set

$$E^{\otimes_\phi n} = E \otimes_\phi \cdots \otimes_\phi E \qquad (n \text{ times }).$$

Let $(E, \phi, \delta, Q)$ be a left or right counital $A$-coring. We defin an element $\omega$ of $\mathcal{L}_A(E^{\otimes_\phi 4}, E^{\otimes_\phi 2})$ by

$$\omega = \{t(I_E \otimes_\phi Q) \otimes_\phi I_E\}(I_E \otimes_{\phi \otimes \iota} \delta^* \otimes_\phi I_E).$$

Then we have $\omega(\omega \otimes_{\phi\otimes\iota} I) = \omega(I \otimes_{\phi\otimes\iota} \omega)$. Therefore we can difine a product on $E \otimes_\phi E$ by $xy = \omega(x \otimes_{\phi\otimes\iota} y)$. Then $E \otimes_\phi E$ is an algebra over $\mathbb{C}$. Note that we have

$$(\xi_1 \otimes_\phi \xi_2)(\eta_1 \otimes_\phi \eta_2) = (\xi_1 Q(\xi_2 \eta_1)) \otimes_\phi \eta_2.$$

**Definition 3.2.** We say that $\delta$ and $Q$ are compatible if the following equation holds; $\delta(\xi\eta) = \delta(\xi)\delta(\eta)$ for every $\xi$, $\eta \in E$.

**Example 3.3** ([13]). Let $1 \in A_0 \subset A_1$ be an inclusion of $C^*$-algebras and let $P_1 : A_1 \to A_0$ be a faithful positive conditional expectation of index finite type. Let $\{u_i, u_i^*; i = 1, \cdots, N\}$ be a quasi-basis of $P_1$. Let $E_1 = A_1$ be a Hilbert $A_0$-module with the $A_0$-valued inner product defined by $< a, b >= P_1(a^*b)$. Let $\phi_1 : A_1 \to \mathcal{L}_{A_0}(E_1)$ be a $*$-homomorphism defined by $\phi_1(a)b = ab$. We denote by $\phi_0$ the restriction of $\phi_1$ to $A_0$. Define $\delta \in \mathcal{L}_{A_0}(E_1, E_1 \otimes_{\phi_0} E_1)$ by $\delta(\xi) = \sum_{i=1}^N (\xi u_i) \otimes_{\phi_0} u_i^*$. The product on $E_1$ induced by $\delta$ agrees with the product on $A_1$. Then $(E_1, \phi_0, \delta, P_1)$ is a compatible counital $A$-coring.

**Example 3.4.** Let $G$ be a finite groupoid. Set $A = C(G^{(0)})$ and $E = C(G)$. Then $E$ is a right $A$-module with the right $A$-action defined by $(\xi a)(x) = \xi(x)a(s(x))$ for $\xi \in E$, $a \in A$ and $x \in G$. We define an $A$-valued inner product of $E$ by

$$< \xi, \eta > (u) = \sum_{g \in G_u} \overline{\xi(g)}\eta(g)$$

for $\xi$, $\eta \in E$ and $u \in G^{(0)}$, where $G_u = s^{-1}(u)$ for $u \in G^{(0)}$. Then $E$ is a Hilbert $A$-module. Define $*$-homomorphisms $\phi$ and $\psi$ of $A$ to $\mathcal{L}_A(E)$ by $(\phi(a)\xi)(x) = a(r(x))\xi(x)$ and $\psi(a) = \xi a$ respectively for $a \in A$, $\xi \in E$ and $x \in G$. Note that we have $E \otimes_\psi E = C(G^2(ss))$ and $E \otimes_\phi E = C(G^{(2)})$, where $G^2(ss) = \{(g, h) \in G^2;\ s(g) = s(h)\}$. Let $W \in \mathcal{L}_A(E \otimes_\psi E, E \otimes_\phi E)$ be the MUO defined by $(W\xi)(g, h) = \xi(h, gh)$. Define an element $a_0 \in A$ by $a_0(u) = |G_u|^{-1/2}$ and define an element $\xi_0 \in E$ by $\xi_0(g) = a_0(s(g))$. Then $\xi_0$ satisfies Properties E1 and E2. Note that we have $||\xi_0|| = 1$. Define an element $\eta_0 \in E$ by $\eta_0 = \chi_{G^{(0)}} a_0^{-1}$. Define operators $Q_{\eta_0}$, $Q_{\xi_0} : E \to A$ by $Q_{\eta_0}(\xi) =< \eta_0, \xi >$ and $Q_{\xi_0}(\xi) =< \xi_0, \xi >$ respectively. Then $(E, \phi, \delta_{\xi_0}, Q_{\eta_0})$ is a compatible counital $A$-coring. The product on $E$ induced by $\delta_{\xi_0}$ is of the form $\xi\eta = (\xi * \eta)a_0$, where $\xi * \eta$ is the convolution product on $C(G)$. We also have two compatible right counital $A$-corings $(E, \psi, \delta_{\xi_0}, Q_{\xi_0})$

and $(E, \psi, \delta_{\xi_0}, Q_{\eta_0})$. Two products on $E \otimes_\psi E$ associated with above right counital $A$-corings are different.

## 4. Coring structures associated with inclusions of $C^*$-algebras

Let $1 \in A_0 \subset A_1$ be an inclusion of $C^*$-algebras and let $P_1 : A_1 \to A_0$ be a faithful positive conditional expectation of index-finite type with a quasi-basis $\{u_i, u_i^*\}_{i=1}^N$. Let $E_1$, $\phi_1$ and $\phi_0$ be as in Example 3.3. Set $E_2 = E_1 \otimes_{\phi_0} E_1$ and define a $*$-homomorphism $\phi_2 : A_1 \to \mathcal{L}_{A_0}(E_2)$ by $\phi_2 = \phi_1 \otimes \iota$. Define a $C^*$-algebra $A$ by $A = \mathcal{L}_{A_0}(E_1, \phi_1)$ and a Hilbert $A$-module $E$ by

$$E = \mathcal{L}_{A_0}((E_1, \phi_1), (E_2, \phi_2)),$$

that is, $E$ is the set of elements $x \in \mathcal{L}_{A_0}(E_1, E_2)$ such that $x\phi_1(a) = \phi_2(a)x$ for all $a \in A$. The $A$-valued inner product on $E$ is defined by $< x, y >= x^*y$. We define $*$-homomorphisms $\phi$ and $\psi$ of $A$ to $\mathcal{L}_A(E)$ by $\phi(a)x = (a \otimes_{\phi_0} I)x$ and $\psi(a)x = (I \otimes_{\phi_0} a)x$ respectively. We suppose that there exists an MUO $W \in \mathcal{L}_A(E \otimes_\psi E, E \otimes_\phi E)$ such that $V^*\widetilde{V} = W \otimes_i I_{E_1}$, where $V : E \otimes_\phi E \otimes_i E_1 \to E_3$ and $\widetilde{V} : E \otimes_\psi E \otimes_i E_1 \to E_3$ are operators defined in [8]. As for sufficient conditions for $W$ to exist, see [7] and [8]. Define an element $x_0 \in E$ by $x_0(\xi) = \xi \otimes_{\phi_0} 1$. Then $x_0$ satisfies Properties E1 and E2. Note that we have $\|x_0\| = 1$. Define an element $\widetilde{y_0} \in E$ by

$$\widetilde{y_0}(\xi) = \sum_{i=1}^N (\xi u_i) \otimes_{\phi_0} u_i^*.$$

Note that we have $\widetilde{y_0}^*(\xi \otimes_{\phi_0} \eta) = \xi\eta$, where $\xi\eta$ is the product on $A_1$. Define $Q_{x_0}, Q_{\widetilde{y_0}} \in \mathcal{L}_A(E, A)$ by $Q_{x_0}(x) =< x_0, x >$ and $Q_{\widetilde{y_0}}(x) =< \widetilde{y_0}, x >$ respecively. Then we have the following theorem.

**Theorem 4.1.** (1) *$(E, \phi, \delta_{x_0}, Q_{x_0})$ is a compatible right counital $A$-coring.*

(2) *Suppose that there esist elements $(v_i, w_i) \in E \times E$ $(i = 1, \cdots K)$ such that*

$$\widehat{\delta}_{x_0}(\widetilde{y_0}) = \sum_{i=1}^K v_i \otimes_\psi w_i.$$

*Then $(E, \psi, \widehat{\delta}_{x_0}, Q_{\widetilde{y_0}})$ is a compatible counital A-coring.*

## 5. Coring structures on the set of compact operators

Let $H$ be an infinite-dimensional separable Hilbert space. We consider $H$ to be a Hilbert $\mathbb{C}$-module, in particular the inner product is linear in the second variable. We denote by $A$ the $C^*$-algebra $\mathcal{K}(H)$ of compact operators on $H$. Let $E$ be a Hilbert $A$-module $\mathcal{K}(H, H \otimes H)$. The right action of $A$ on $E$ is defined by $(xa)(\xi) = x(a(\xi))$ for $x \in E$, $a \in A$ and $\xi \in H$ and the $A$-valued inner product of $E$ is defined by $< x, y >= x^* y$ for $x$, $y \in E$. Define $*$-homomorphisms $\phi$ and $\psi$ of $A$ to $\mathcal{L}_A(E)$ by $\phi(a)x = (a \otimes I_H)x$ and $\psi(a)x = (I_H \otimes a)x$ for $a \in A$ and $x \in E$ respectively. We denote by $F$ the Hilbert $A$-module $\mathcal{K}(H, H \otimes H \otimes H)$. The right action of $A$ on $F$ and the $A$-valued inner product of $F$ are defined by the same formulas as those in $E$. There exist unitary operators $M \in \mathcal{L}_A(E \otimes_\phi E, F)$ and $\widetilde{M} \in (E \otimes_\psi E, F)$ such that

$$M(x \otimes_\phi y) = (x \otimes I_H)y,$$

$$\widetilde{M}(x \otimes_\psi y) = (I_H \otimes x)y$$

for $x$, $y \in E$ respectively. Define $W = M^{-1}\widetilde{M}$. Then we have the following:

**Theorem 5.1.** *The operator $W$ is the unique multiplicative unitary operator in $\mathcal{L}_A(E \otimes_\psi E, E \otimes_\phi E)$.*

Now we introduce a coring structure on $(E, \phi, \psi)$. Recall that an approximate unit $\{u_n\}_{n=1}^\infty$ of $A$ is said to be increasing if $u_n \geq 0$ and $u_{n+1} \geq u_n$ for every $n$.

**Definition 5.2.** Let $\delta$ be a coproduct of $(E, \phi)$. For $n = 1, 2, \cdots$, let $Q_n$ be an element of $\mathcal{L}_A(E, A)$ such that $Q_n(\phi(a)x) = aQ_n(x)$ for $a \in A$ and $x \in E$ and let $\{u_n\}_{n=1}^\infty$ be an increasing approximate unit of $A$ such that $u_1 \neq 0$ and $u_n \neq u_{n+1}$ for every $n$. Then $(\delta, \{Q_n\}_{n=1}^\infty, \{u_n\}_{n=1}^\infty)$ is called a coring structure on $(E, \phi, \psi)$ if it satisfies the following equations for every $n$;

$$t(I_E \otimes_\phi Q_n)\delta = t'(Q_n \otimes_\phi I_E)\delta = \psi(u_n),$$

$$Q_n \psi(u_n) = Q_n.$$

Then $\{Q_n\}$ is called an approximate counit.

Let $T$ be an element of $\mathcal{L}(H, H \otimes H)$. We will say that $T$ has Property D if it satisfies the following conditions:

(i) $(T \otimes I_H)T = (I_H \otimes T)T$.

(ii) There exists a family $\{K_n\}_{n=1}^{\infty}$ of mutually orthogonal non-trivial finite-dimensional subspaces of $H$ such that $H = \oplus_{n=1}^{\infty} K_n$ and there exists a complete orthonormal basis $\{e_{k_{n-1}+1}, \cdots, e_{k_n}\}$ of $K_n$ for $n = 1, 2, \cdots$, where $k_0 = 0$, such that, if we set $\lambda_{j,\ell}^i =< e_j \otimes e_\ell, Te_i >$, then $\{\lambda_{j.\ell}^i\}$ satisfies the following conditions;

(a) for $i = k_{n-1} + 1$, $\lambda_{i,i}^i \neq 0$ and $\lambda_{j,\ell}^i = \lambda_{\ell,j}^i = 0$ for every $j \in \mathbb{N}$ and $\ell = k_m + 1$ $(m = 0, 1, 2, \cdots)$ except for $j = \ell = i$,

(b) if dim $K_n \geq 2$, for $i = k_{n-1} + 2, \cdots, k_n$,

$$\lambda_{i,k_{n-1}+1}^i = \lambda_{k_{n-1}+1,i}^i = \lambda_{k_{n-1}+1,k_{n-1}+1}^{k_{n-1}+1},$$

and $\lambda_{j,\ell}^i = \lambda_{\ell,j}^i = 0$ for every $j \in \mathbb{N}$ and $\ell = k_m + 1$ $(m = 0, 1, 2, \cdots)$ except for $(j, \ell) = (i, k_{n-1} + 1)$.

Then we have the following theorem:

**Theorem 5.3.** *There exists a one-to-one correspondence between the set of coring structures $(\delta, \{Q_n\}, \{u_n\})$ on $(E, \phi, \psi)$ and the set of elements $(T, \{K_n\}, \{e_{k_{n-1}+1}\})$ which satisfy Property D. The correspondence is given as follows: If $(T, \{K_n\}, \{e_{k_{n-1}+1}\})$*

*has Property D, set*

$$H_n = \oplus_{i=1}^n K_i,$$

$$\xi_n = \sum_{i=1}^n \eta_i \in H_n, \quad \textit{where } \eta_i = (\overline{\lambda_{k_{i-1}+1,k_{i-1}+1}^{k_{i-1}+1}})^{-1} e_{k_{i-1}+1} \in K_i,$$

*define* $f_n \in H^*$ *by* $f_n(\xi) =< \xi_n, \xi >$, *then* $u_n \in \mathcal{K}(H)$ *is the projection onto* $H_n$ *and* $\delta$ *and* $Q_n$ *are given by the following equations;*

$$\delta(x) = M^{-1}(I_H \otimes T)x,$$

$$Q_n(x) = (I_H \otimes f_n)x.$$

*Question.* Suppose that $T$ has Property D. Does $T$ determine $\{K_n\}$ and $\{e_{k_{n-1}+1}\}$ uniquely?

The following theorem shows the relation between the coring structures and the multiplicative unitary operator $W$ defined above:

**Theorem 5.4.** *Let* $(\delta, \{Q_n\}, \{u_n\})$ *be a coring structure on* $(E, \phi, \psi)$ *and let* $T$ *be the operator which corresponds to* $(\delta, \{Q_n\}, \{u_n\})$ *by Theorem 5.3. Put* $x_n = Tu_n$. *Then* $x_n$ *is an element of* $E$ *and satisfies Property E1. Let* $\delta_n$ *be the coproduct of* $(E, \phi)$ *defined by*

$$\delta_n(x) = W(x_n \otimes_\psi x).$$

*Then the following equation holds;*

$$\delta = \lim_{n \to \infty} \delta_n$$

*with respect to the strict topology on* $\mathcal{L}_A(E, E \otimes_\phi E)$.

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DEPARTMENT OF APPLIED MATHEMATICS, FACULTY OF SCIENCE, OSAKA WOMEN'S UNIVERSITY, SAKAI CITY, OSAKA 590-0035, JAPAN

*E-mail address*: ouchi@appmath.osaka-wu.ac.jp